D1: Linear Recurrences

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K field (e.g. $K=\mathbb{Q}$)

$(a_n)_{n\geq 0}$ is an LRS if $\exists c_1,\dots,c_d \in K$ s.t.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} \quad \forall n \geq d \qquad (*)$$

Prop 1.1: TFAE

(a) $(a_n)_{n\geq 0}$ is an LRS

(b) The generating function $f = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$ (formal power series) is rational, i.e., $f = P/Q$ with $P, Q \in K[x]$, $Q(0)=1$

(c) $\exists d \geq 0$ $\exists u, v \in K^d$, $A \in K^{d\times d}$ s.t.

$$a_n = u^T A^n v \quad \forall n \geq 0.$$

Proof: (a) $\Rightarrow$ (b)

$$f = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=d}^{\infty} \sum_{i=1}^{d} c_i a_{n-i} x^n + \underbrace{\sum_{n=0}^{d-1} a_n x^n}_{=:P_0 \in K[x]}$$

$$= \sum_{i=1}^{d} c_i \underbrace{\left(\sum_{n=d}^{\infty} a_{n-i} x^{n-i} \right)}_{= f - \underbrace{\sum_{n=0}^{d-i-1} a_n x^n}_{= P_i(x)}} x^i + P_0 = f\tilde{Q} + R \text{ with } R \in K[x],$$

$\deg R < d$, and $\tilde{Q} = \sum_{i=1}^{d} c_i x^i$

$$\Rightarrow f(1-\tilde{Q}) = R \Rightarrow f = P/Q, \quad Q = 1-\tilde{Q}.$$

(b) $\Rightarrow$ (a) Say $Q = 1 - c_1 x - \dots - c_d x^d$, $c_d \neq 0$

$P = Qf$

$$\Rightarrow P = \sum_{n=0}^{\infty} (a_n - a_n c_1 x - \dots - a_n c_d x^d) x^n$$

$$\Rightarrow P = \sum_{n=d}^{\infty} (a_n - a_{n-1} c_1 - \dots - a_{n-d} c_d) x^n + R, \quad R \in K[x], \deg R < d$$

$\Rightarrow$ if $n \geq d' := \max\{\deg P + 1, d\}$, then

$$a_n = \sum_{i=1}^{d} c_i a_{n-i} = \sum_{i=1}^{d'} c_i a_{n-i} \quad \text{with} \quad c_{d+1} = \dots = c_{d'} = 0.$$

(a) $\Rightarrow$ (c) Take

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ c_d & \cdots & \cdots & c_1 \end{pmatrix}, \quad v = \begin{pmatrix} a_0 \\ \vdots \\ a_{d-1} \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

companion matrix

Then $A^n v = \begin{pmatrix} a_n \\ \vdots \\ a_{n+d-1} \end{pmatrix}$, $u A^n v = a_n$

(c) $\Rightarrow$ (d) If $P(x) := \det(xI - A) = x^d - c_1 x^{d-1} - \dots - c_d$, then $P(A) = 0$ by Cayley-Hamilton.

$$\Rightarrow \underbrace{u^T A^n v}_{= a_n} = c_1 \underbrace{u^T A^{n-1} v}_{= a_{n-1}} + \dots + c_d \underbrace{u^T A^{n-d} v}_{= a_{n-d}} \quad \forall n \geq d.$$

$\square$

Obs: Can take $c_d \neq 0$ in (*)

$\Leftrightarrow$ Can take $\det(A) \neq 0$ in (c) [since char. poly of the companion matrix is $x^d - c_1 x^{d-1} - \dots - c_{d-1} x - c_d$ (exc.)], i.e. $A \in GL_d(K)$

$\Leftrightarrow$ Can take $\deg P < \deg Q$ in (b)

Def: An LRS is strict if we can take $c_d \neq 0$ in (*)

Prop 1.2: For any LRS $(a_n)_{n \geq 0}$, there exists a unique strict LRS $(b_n)_{n \geq 0}$ s.t. $a_n = b_n$ for all sufficiently large n.

Proof: If $f = \sum_{n=0}^{\infty} a_n x^n = P/Q$, then $f = R + \frac{P'}{Q}$ with $R, P' \in K[x]$, $\deg P' < \deg Q$ (polynomial division). Take $(b_n)_{n \geq 0}$ s.t. $\frac{P'}{Q} = \sum_{n=0}^{\infty} b_n x^n$.

If $(\tilde{b}_n)_{n \geq 0}$ is another such strict LRS, then

If $(b_n)_{n\geq 0}$ is another such strict LRS, then

$$\sum_{n=0}^{\infty} \tilde{b}_n x^n = \tilde{P}/\tilde{Q}, \quad \deg(\tilde{P}) < \deg(\tilde{Q}), \quad f = \tilde{R} + \tilde{P}/\tilde{Q}, \quad \tilde{R} \in K[x]$$

$$\Rightarrow \frac{RQ+P'}{Q} = \frac{\tilde{R}\tilde{Q}+\tilde{P}}{\tilde{Q}} \Rightarrow RQ\tilde{Q} + P'\tilde{Q} = \tilde{R}Q\tilde{Q} + \tilde{P}Q$$

$$\overset{\text{poly.div.}}{\underset{\text{by } Q\tilde{Q}}{\Longrightarrow}} R = \tilde{R}, \quad P'\tilde{Q} = \tilde{P}Q \quad \text{so} \quad P'/Q = \tilde{P}/\tilde{Q}, \text{ hence } b_n = \tilde{b}_n \ \forall n \geq 0. \qquad \square$$

Remark: Every LRS satisfies a recurrence (*) with minimal d. The corresponding $c_1, \dots, c_d$ are unique.

[Say, also $a_n = c_1' a_{n-1} + \dots + c_d' a_{n-d}$ for $n \geq d$ with $(c_1, \dots, c_d) \neq (c_1', \dots, c_d')$.
Subtraction gives $(c_1' - c_1) a_{n-1} + \dots + (c_d' - c_d) a_{n-d} = 0$. Let i be minimal with $c_i' - c_i \neq 0$. Dividing by $c_i' - c_i$ gives a shorter recurrence $\lightning$]

Then $x^d - c_1 x^{d-1} - \dots - c_{d-1} x - c_d$ is the minimal polynomial of $(a_n)_{n\geq 0}$, its roots are the eigenvalues.

Prop 1.3 If char $K = 0$, all eigenvalues of $(a_n)_{n\geq 0}$ are in K (e.g. $K = \bar{K}$) (say $\lambda_1, \dots, \lambda_s$ pw. distinct), and $(a_n)_{n\geq 0}$ is strict, then

$$\forall n \geq 0: \quad a_n = P_1(n)\lambda_1^n + \dots + P_s(n)\lambda_s^n.$$

← exponential polynomial

Proof: Let $Q = x^d - c_1 x^{d-1} - \dots - c_d$ be the min. poly of $(a_n)_{n\geq 0}$, $c_d \neq 0$.
By [P1.1, (a) → (b)], $f = \sum_{n=0}^{\infty} a_n x^n = \frac{P}{\tilde{Q}}$ with

$\tilde{Q} = 1 - c_1 x - c_2 x^2 - \dots - c_d x^d = x^d Q(\frac{1}{x})$ the reciprocal polynomial, with roots $1/\lambda_1, \dots, 1/\lambda_s$. Let $e_i \geq 1$ be the multiplicity of λ_i in Q.

$$\overset{\text{Partial Fraction}}{\underset{\text{Decomposition}}{\Longrightarrow}} f = \sum_{i=1}^{s} \sum_{j=1}^{e_i} \frac{b_{ij}}{(x - \frac{1}{\lambda_i})^j}, \quad b_{ij} \in K.$$

$$\text{Now } \frac{1}{(x-\frac{1}{\lambda_i})^j} = \pm \lambda_i^j \frac{1}{(1-\lambda_i x)^j} = \pm \lambda_i^j \sum_{n=0}^{\infty} \underbrace{\binom{n+j-1}{j-1}}_{= \frac{(n+j-1)\cdot\ldots\cdot(n+1)}{(j-1)!}, \text{ polynomial in } n} \lambda_i^n x^n. \qquad \square$$

<u>Remark:</u> (1) Conversely, every exp. poly defines a strict LRS

(2) The exp. poly repr is unique. If $K = \overline{K}$, there is a bij

$\{\text{strict LRS}\} \leftrightarrow \{\text{exp. polys}\}$.